Lecture 12. Laplace Transformation

- Description of AC signals
- Phasors
- Complex numbers
- Step & Delta functions
- Laplace Transform
Description of AC Signals

• Period: \( T = \frac{2\pi}{\omega} = \frac{1}{f} \)
  – Time to complete one cycle

• Frequency: \( f = \frac{1}{T} \)
  – # of cycles per second (Hertz, Hz)

• Angular frequency: \( \omega = 2\pi f \)
  – Radians per second

• Amplitude: \( A \)
  – For example, could be volts or amps
Phasors: Representation of AC signal

- A phasor is a complex number that represents the magnitude and phase angle of a sinusoidal signal:

\[ X_M \cos(\omega t + \theta) \]

\[ X = X_M \angle \theta \]
Complex Numbers

- \( x \) is the real part
- \( y \) is the imaginary part
- \( z \) is the magnitude
- \( \theta \) is the phase

![Diagram of Complex Numbers]

imaginary axis

real axis
Complex Numbers

- Polar coordinates: $A = z \angle \theta$
- Rectangular coordinates: $A = x + jy$
- Polar $\leftrightarrow$ rectangular conversion relations:

\[
\begin{align*}
x &= z \cos \theta \\
z &= \sqrt{x^2 + y^2} \\
y &= z \sin \theta \\
\theta &= \tan^{-1} \frac{y}{x}
\end{align*}
\]
Arithmetic with Complex Numbers

- To compute phasor voltages and currents, we need to be able to perform basic computations with complex numbers
  - Addition
  - Subtraction
  - Multiplication
  - Division
- Appendix A has a review of complex numbers
Complex Number
Addition and Subtraction

- Addition is most easily performed in rectangular coordinates:
  \[ A = x + jy \quad \text{and} \quad B = z + jw \]
  \[ A + B = (x + z) + j(y + w) \]

- Subtraction is also most easily performed in rectangular coordinates:
  \[ A - B = (x - z) + j(y - w) \]
Complex Number
Multiplication and Division

• Multiplication is most easily performed in polar coordinates:

\[ A = A_M \angle \theta \quad \text{and} \quad B = B_M \angle \phi \]

\[ A \times B = (A_M \times B_M) \angle (\theta + \phi) \]

• Division is also most easily performed in polar coordinates:

\[ A / B = (A_M / B_M) \angle (\theta - \phi) \]
Examples

- Convert to polar: $3 + j4$ and $-3 - j4$
- Convert to rectangular: $2 \angle 45^\circ$ & $-2 \angle 45^\circ$
- Add: $3 \angle 30^\circ + 5 \angle 20^\circ$
- Determine the complex conjugate of the phasor: $A = z \angle \theta$
Singularity Functions

- *Singularity functions* are either not finite or don't have finite derivatives everywhere.
- The two singularity functions of interest are:
  1. *unit step* function, $u(t)$
     and its construct: the gate function
  2. *delta* or *unit impulse* function, $\delta(t)$
     and its construct: the sampling function
Unit Step Function, $u(t)$

- The *unit step* function, $u(t)$
  - Mathematical definition
    $$ u(t) = \begin{cases} 
      0 & t < 0 \\
      1 & t > 0 
    \end{cases} $$
  - Graphical illustration
Extensions of Unit Step Function

• A more general unit step function is $u(t-a)$

$$u(t - a) = \begin{cases} 
0 & t < a \\
1 & t > a 
\end{cases}$$

• The gate function can be constructed from $u(t)$
  – a rectangular pulse that starts at $t=\tau$ and ends at $t=\tau+T$:
    $u(t-\tau) - u(t-\tau-T)$
  – like an on/off switch
Delta or Unit Impulse Function, $\delta(t)$

- The *delta* or *unit impulse* function, $\delta(t)$
  - Mathematical definition (non-purist version)
    \[
    \delta(t - t_0) = \begin{cases} 
    0 & t \neq t_0 \\
    1 & t = t_0 
    \end{cases}
    \]
  - Graphical illustration

![Graphical illustration of the delta function](image-url)
Extensions of the Delta Function

- An important property of the unit impulse function is its **sampling property**
  - Mathematical definition (non-purist version)

\[ f(t) \delta(t - t_0) = \begin{cases} 
0 & t \neq t_0 \\
 f(t_0) & t = t_0 
\end{cases} \]
Laplace Transform

• Applications of the Laplace transform
  – solve differential equations (both ordinary and partial)
  – application to RLC circuit analysis

• Laplace transform converts differential equations in the time domain to algebraic equations in the frequency domain, thus three important processes:
  1. transformation from the time to frequency domain
  2. manipulate the algebraic equations to form a solution
  3. inverse transformation from the frequency to time domain (we’ll wait to the next class time for this)
Definition of Laplace Transform

- Definition of the unilateral (one-sided) Laplace transform

\[ L[f(t)] = F(s) = \int_{0}^{\infty} f(t) e^{-st} \, dt \]

where \( s = \sigma + j\omega \) is the complex frequency, and \( f(t) = 0 \) for \( t < 0 \)

- The inverse Laplace transform requires a course in complex variable analysis (e.g., MAT 461)
## A Useful Table

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>1</td>
</tr>
<tr>
<td>$u(t) {a \text{ constant}}$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$e^{-at}$</td>
<td>$\frac{1}{s + a}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$t e^{-at}$</td>
<td>$\frac{1}{(s + a)^2}$</td>
</tr>
</tbody>
</table>
# Some Laplace Transform Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Scaling</strong></td>
<td>$A f(t)$</td>
<td>$A F(s)$</td>
</tr>
<tr>
<td><strong>Linearity</strong></td>
<td>$f_1(t) \pm f_2(t)$</td>
<td>$F_1(s) \pm F_2(s)$</td>
</tr>
<tr>
<td><strong>Time Scaling</strong></td>
<td>$f(a \cdot t)$</td>
<td>$\frac{1}{a} F\left(\frac{s}{a}\right)$</td>
</tr>
<tr>
<td><strong>Time Shifting (delay)</strong></td>
<td>$f(t-t_0) \ u(t-t_0)$</td>
<td>$e^{-s t_0} F(s)$</td>
</tr>
<tr>
<td><strong>Frequency Shifting</strong></td>
<td>$e^{-a \cdot t} f(t)$</td>
<td>$F(s+a)$</td>
</tr>
<tr>
<td><strong>Time Domain Differentiation</strong></td>
<td>$\frac{d f(t)}{dt}$</td>
<td>$s F(s) - f(0)$</td>
</tr>
<tr>
<td><strong>Frequency Domain Differentiation</strong></td>
<td>$t \ f(t)$</td>
<td>$-\frac{d F(s)}{ds}$</td>
</tr>
<tr>
<td><strong>Time Domain Integration</strong></td>
<td>$\int_0^t f(\tau) \ d\tau$</td>
<td>$\frac{1}{s} F(s)$</td>
</tr>
<tr>
<td><strong>Convolution</strong></td>
<td>$\int_0^t f_1(\tau) f_2(t - \tau) \ d\tau$</td>
<td>$F_1(s) \ F_2(s)$</td>
</tr>
</tbody>
</table>
Class Examples

• TBA